Solutions Midterm Exam Complex Analysis 2012

1. a) $f(z) = u(x,y) + iv(x,y)$ with $u(x,y) = x^2+y^2$ and $v(x,y) = 2xy$. The partial derivatives exist everywhere and are given by $\frac{\partial u}{\partial x} = 2x$, $\frac{\partial u}{\partial y} = 2y$, $\frac{\partial v}{\partial x} = 2y$, $\frac{\partial v}{\partial y} = 2x$

The Cauchy-Riemann equations are:
$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$; $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. Clearly these hold in a point $z = x+iy$ if and only if $y=0$.

Conclusion: there does not exist any point $z = x+iy$ with a neighbourhood on which $f(z)$ is differentiable!

b) I claim that $f(z)$ is differentiable in every point $z = x+iy$ with $y=0$, so on the real axis. Indeed, take such point. Then

1. the partial derivatives of u and v exist in a neighbourhood of that point
2. They are continuous in the point z
3. The Cauchy-Riemann equations hold in z.

These 3 conditions are sufficient to conclude that f is differentiable in z.

2. a) The function $\cos \zeta$ is analytic on $\mathbb{C}$, so by the Cauchy integral formula, for all z inside the circle Γ we have $\cos z = \frac{1}{2\pi i} \int_\Gamma \frac{\cos \zeta}{\zeta - z} d\zeta$.

Thus: for z inside Γ we have that $F(z) = \cos z$
$\Longrightarrow F(\pi) = \cos \pi = -1$.

Next, $F(0) = \frac{1}{2\pi i} \int_\Gamma \frac{\cos \zeta}{\zeta} d\zeta$. The function $f(\zeta) = \frac{\cos \zeta}{\zeta}$ is analytic on the domain $\{z \mid \operatorname{Re} z > 0\}$ and Γ is a closed contour in that domain:

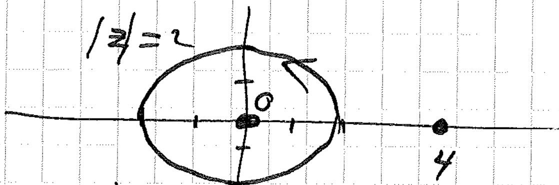
Hence $F(0) = \frac{1}{2\pi i} \int_\Gamma \frac{\cos \zeta}{\zeta} d\zeta = 0$

b) To be shown: $\lim_{z \to \frac{1}{2}\pi} F(z) = F(\frac{1}{2}\pi)$. First note that by definition $F(\frac{1}{2}\pi) = 0$ since $\frac{1}{2}\pi$ is <u>on</u> Γ.
So we need to show that $F(z) \to 0$ as $z \to \frac{1}{2}\pi$.
There are 3 possibilities:

1. $z \to \frac{1}{2}\pi$ along a path <u>inside</u> Γ. In that case $F(z) = \cos z \to \cos \frac{1}{2}\pi = 0$ as desired
2. $z \to \frac{1}{2}\pi$ along a path <u>outside</u> Γ. In that case $F(z) = 0 \to 0$ as desired
3. $z \to \frac{1}{2}\pi$ along a path <u>on</u> Γ. In that case $F(z) = 0 \to 0$ as desired.

c) <u>NO</u>:
Take any point z_0 on Γ such that $\cos(z_0) \neq 0$.
Since $F(z) = \cos z$ for z inside Γ, we see that if $z \to z_0$ along a path inside Γ, then $F(z) = \cos(z) \to \cos(z_0) \neq 0$, while $F(z_0) = 0$ by definition. Conclusion: $\lim_{z \to z_0} F(z) \neq F(z_0)$.

3) Define the function $f(z) = \frac{\sin z}{z-4}$. This function is analytic on and inside the circle C
0 lies <u>inside</u> Γ.



By Cauchy's integral formula we have:

$$f'(0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z^2}\, dz .$$

Thus: $$\int_C \frac{\sin z}{z^2(z-4)}\, dz = 2\pi i\, f'(0) . \quad (*)$$

$$f'(z) = \frac{(\cos z)(z-4) - \sin z}{(z-4)^2} \Rightarrow f'(0) = \frac{-4}{16} = -\frac{1}{4}$$

Conclusion: $(*) = -\frac{1}{2}\pi i$

4) a) $f(z) = \text{Re}\, f(z) + i\, \text{Im}\, f(z)$ thus

$$e^{f(z)} = e^{\text{Re}\, f(z)} \cdot e^{i\, \text{Im}\, f(z)}.$$

Also we have

$$\left| e^{f(z)} \right| = \left| e^{\text{Re}\, f(z)}\, e^{i\, \text{Im}\, f(z)} \right| = e^{\text{Re}\, f(z)}$$

Since $\text{Re}\, f(z) \leq M$ for all $z \in \mathbb{C}$ we conclude:

$$|g(z)| = \left| e^{f(z)} \right| = e^{\text{Re}\, f(z)} \leq e^{M} \quad \forall z.$$

So $g(z)$ is bounded

b) Claim: $g(z)$ is a constant function.
Indeed: $g(z)$ is entire (as the composition of $f(z)$ and e^{w}) and bounded. Apply then Liouville's theorem.
So: $g(z)$ is constant. But then also $f(z)$ must be constant

5) $f(z) = z^3 \sin 2z$

a) $f(z)$ is analytic on $\mathbb{C}$, so take $D = \mathbb{C}$.

b) $\sin 2z = z - \frac{1}{3!}(2z)^3 + \frac{1}{5!}(2z)^5 - \ldots\ldots$

$$\Rightarrow f(z) = z^3 \sin 2z = z^4 - \frac{8}{3!} z^6 + \frac{2^5}{5!} z^8 - \ldots\ldots$$

c) Is convergent for all $z \in \mathbb{C}$ since $z^3 \sin 2z$ is analytic on $\mathbb{C}$.